Transpn. Res.-B Vol. 18B, No. 4/5, pp. 289–299, 1984
Printed in the U.S.A.

0191-2615/84 $3.00 + .00
Pergamon Press Ltd.

ESTIMATION OF TRIP MATRICES FROM TRAFFIC COUNTS AND SURVEY DATA: A GENERALIZED LEAST SQUARES ESTIMATOR

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(*Received* 5 *November* 1982; *in revised form* 28 *July* 1983)

Abstract—Methods commonly used for estimating origin–destination (O–D) matrices can be divided into three categories: direct sample estimation, model estimation and estimation from traffic counts. In this paper a generalized least squares estimator of the O–D matrix is proposed combining direct or model estimators with traffic counts via an assignment model. The presence of measurement errors and time variability in the observed flows is explicitly considered. A special case is also presented in which the flows are assumed to be deterministically known. For the proposed estimators, means and dispersion matrices are expressed in function of the possible bias in the direct or model estimators and assignment model misspecification. The variance of the O–D matrix obtained with the generalized least squares (GLS) estimators is proved to be lower than that obtained with direct or model estimators but, because of possible biases and misspecifications, it is suggested that their performances have to be compared by using risk or generalized mean square error criteria.

1. INTRODUCTION

The Origin–Destination (O–D) trip matrix is a fundamental input for most problems regarding planning and management of transportation systems. In practice, the "true" O–D matrix is seldom, if ever, available and various methods can be used for its estimation. These methods, because the very different contests to which the O–D matrix is relative, from a single intersection to a whole region, are very different among them and can be divided into three categories.

The first group of methods can be defined as *direct sample estimation* of the O–D matrix. Many types of surveys, such as home or destination interviews, roadside interviews, flagging techniques, etc. or combination of them are used depending on the problem at hand; the O–D matrix is estimated by these survey results using one of the sampling theory classical estimators. The variances of the O–D cell values and their covariances depend on the estimator adopted and on the sampling technique used, such as geographic stratification, cluster sampling, etc. and can be obtained by standard sampling theory results, Cochran (1961); Yates (1981). Furthermore, the estimates are often biased because of "non-response" and systematic "measurement" errors (understatement of journeys carried out, etc.).

A second method commonly used can be defined as *model estimation*. The O–D matrix is estimated by applying a system of models that give the number of journeys made with a certain mode during a certain period of time. It should be noted that for the above-mentioned application the demand models are used only as a relation between the variable to be estimated (O–D matrix) and other variables with the aim of obtaining more accurate estimates of the former.

Whatever the specification and the hypotheses underlying the models adopted, their parameters can be estimated on the basis of surveys carried out in the study area, or else models calibrated in analogous situations can be used. In any case it is possible to obtain an estimate of the variances and of the covariances (dispersion matrix) of the estimated parameters and therefore of the deriving O–D matrix. It should be added that it is very likely that models used are more or less misspecified, especially those calibrated in different places and/or time and that, therefore, the model estimator for the O–D matrix is biased.

Finally, the third method is the *estimation of the O–D matrix from traffic flows*. This approach is relatively recent in respect of the other two but it has met with a lot of attention because of the great economic advantages it offers. These are derived from the carrying out of flow measurements instead of the more expensive surveys. Furthermore traffic counts are repeatable so that the evolution of the phenomenon can be followed.

Among the estimators of this third group, a first distinction can be made between methods that use traffic flows for the estimation of the parameters of demand models, by which, in turn, the O-D matrix is obtained, and those that use the flows to estimate the O–D matrix directly. For an analysis of the vast bibliography, the works of Willumsen (1978) and Nguyen (1982) can be referred to.

Here we will make explicit reference to the second approach which estimates the O–D matrix that minimizes a measure of distance, "entropic" or euclidian, from a "target" matrix, given by a model or by an old direct estimate, respecting the constraint that, once assigned to the network, the observed flows are reproduced on some links, Van Zuylen and Willumsen (1980), Gur *et al.* (1980), and Nguyen (1982).

Many of the proposed estimators require that the constraint flows respect consistency conditions. Because of measurement errors and approximation in the network extraction procedure, the observed flows are usually inconsistent so that the application of some "consistency estimation" is required beforehand.

Only recently some statistical aspects of the estimation were considered; Bell (1983) expressed the variance-covariance matrix of the maximum entropy estimator in function of the observed flows dispersion matrix and Maher (1983) proposed a Bayesian estimator for the O–D table in which a multivariate normal distribution was hypothesized both for the trip matrix prior distribution and the observed flows.

Not much work has been done to quantify the effects of the assignment models approximations and of the distance of the "target" matrix from the true one on the resulting estimates. Nor has the comparison of the various estimators statistical performances been considered.

In this paper an estimator for the O–D matrix is proposed that combines direct or model estimators† with traffic counts on some network links by means of an assignment model. The presence of measurement errors and of temporal variability in the observed flow is explicitly considered so as to avoid the problem of preventive evaluation of consistent flows.

A particularization of the estimator to the case in which the observed flows can be considered deterministically known is also presented.

The estimator obtained, which is a generalized least squares (Aitken) estimator, under the hypothesis that the direct or model estimators are unbiased and that misspecification errors of the assignment model are negligible, is the one of minimum variance among all unbiased estimators linear in the starting O–D matrix estimate and in the observed flows and, under normality assumptions, among all unbiased estimators.

Because the starting O–D matrix estimates might be biased and the assignment model misspecified, the proposed estimator itself might be biased and its performances should be compared with those of the direct or model estimators under criteria of risk and/or generalized mean square error. For this reason the mean and the dispersion matrix of the proposed estimator were calculated in terms of the errors in the direct or model estimator of the O–D matrix and in the assignment model; confidence intervals for the obtained estimates can thus be set up. Furthermore, it is possible to obtain utilization domains of the various estimators, or, in other terms, it is possible to evaluate how far the information contained in the flows improves the O–D matrix estimates even when there are errors in the assignment model. Obviously the errors present in the various hypotheses are hardly, if ever, exactly known. However, the "practical" knowledge of the magnitude order can help a better estimation of the O–D matrix

2. STATEMENT OF THE PROBLEM

The values of the true O–D matrix relative to the journeys made between n pairs of centroids with a certain mode‡ during a given period of time are ordered in a column vector $\mathbf{t}$ of dimension $n \times 1$. If $\hat{\mathbf{t}}$ indicates the vector analogously obtained by direct or

†In the following O–D matrix estimates obtained by direct or model estimators will be often referred to as "starting estimates".

‡Here specific reference will be made to car journeys. Analogous results can be obtained for other transport modes.

model estimates of the O–D matrix, it can be posed

$$\hat{\mathbf{t}} = \mathbf{t} + \boldsymbol{\epsilon} \tag{1}$$

where $\boldsymbol{\epsilon}$ is a random vector with mean $\boldsymbol{\mu}$ and dispersion matrix $\mathbf{V}$.

This is the same as saying that, in general, the estimator $\hat{\mathbf{t}}$ of the O–D matrix $\mathbf{t}$ has a bias $\boldsymbol{\mu}$. Obviously the dispersion matrix $\mathbf{V}$ depends on the specific estimator adopted, Cascetta (1983) In the case that $\hat{\mathbf{t}}$ is a direct estimate, the sampling theory results relative to the estimator, e.g. sample mean or ratio estimators, and to the sampling technique, such as geographic stratification or cluster sampling, allows the expression of the variances and covariances at the elements of $\hat{\mathbf{t}}$ in function of the values assumed by the variables in the universe, Cochran (1963), Yates (1981). A very simple example of direct estimator with its dispersion matrix will be given in Section 5.

On the other hand, in the case in which a model estimator is used, it will be a function of a parameter's vector $\hat{\boldsymbol{\omega}}$ estimated on a sample. The probability theory results allow the matrix $\mathbf{V}$ to be expressed, more or less approximately, in terms of the dispersion matrix $\mathbf{H}$ of the parameters estimator used and of the functional form of the model.

As a first approximation, $\mathbf{V}$ could be obtained by expanding the model function $g(\boldsymbol{\omega})$ in a Taylor series around the true value of the parameter's vector $\boldsymbol{\omega}$, Kendall (1973).

When the vector $\hat{\mathbf{t}}$ has been obtained by applying a demand model calibrated in a different situation, the dispersion matrix can still be obtained, but it must be realized that in this case the bias $\boldsymbol{\mu}$ due to the model misspecification can be substantial. Finally it should be mentioned that in the case where the results of the study cover a period of time and the daily fluctuation of the O–D matrix being studied cannot be considered negligible, the temporal variance should be added to the sampling variance of the components of $\hat{\mathbf{t}}$, with the former that can be very high, Van Vliet and Willumsen, (1981).

We will now go on to study the information on the O–D matrix contained in the traffic flows observed on a certain number of links of the network.

It is well known that the flow passing in the time period under study of a certain day on each network link is a linear combination of the elements of the O–D matrix of that day with coefficients comprised between 0 and 1. If we consider m network links this can be expressed as:

$$\mathbf{A}\mathbf{t} = \mathbf{f} \tag{2}$$

where $\mathbf{A}$ is the assignment matrix of dimension $m \times n$, and $\mathbf{f}$ the flows vector. Obviously the elements of matrix $\mathbf{A}$ are not known exactly, but they must be obtained by one of the various assignment models. Some of these models assume that the route choice mechanism and therefore the resulting assignment matrix $\hat{\mathbf{A}}$, is independent of the O–D matrix, proportional models. Others assume that it depends on the O–D matrix in a deterministic way, equilibrium models, Florian and Nguyen (1976), Nguyen (1976) or in a random way, stochastic assignment models Dyal (1971).

Another error source in the $\hat{\mathbf{A}}$ matrix obtained by an assignment model, lies in the schematization of the road network in a graph. The above discussion can be synthetically expressed as:

$$\mathbf{f} = \hat{\mathbf{A}}\mathbf{t} + \boldsymbol{\delta} \tag{3}$$

where $\boldsymbol{\delta}$ is generally a random vector.

The random part of $\boldsymbol{\delta}$ is present in the case in which the elements of $\hat{\mathbf{A}}$ are obtained with a model using an O–D matrix estimate $\hat{\mathbf{t}}$; in this case, however, it is very complicated to express the functional dependence of $\hat{\mathbf{A}}$ on $\hat{\mathbf{t}}$. Because of this difficulty, in the following explicit reference will be made to the case in which the assignment model gives $\hat{\mathbf{A}}$ independently of $\hat{\mathbf{t}}$, i.e. proportional assignment. In this case the random part vanishes and the vector $\boldsymbol{\delta}$ represents the sum of the specification error of the assignment model and of the bias deriving from the network extraction process.

Up to now the observed flows were considered to be the "true" ones. In reality this is not the case because of the presence of measurement errors and of time variation and covariation if the flows are measured over several days.

The eventual bias of the observed flow vector $\hat{\mathbf{f}}$ can be ignored with respect to $\boldsymbol{\delta}$, so that it can be assumed to be unbiased with mean $\mathbf{f}$ and dispersion matrix $\mathbf{W}$ which, in lack of more specific information, is often considered to be diagonal with variances equal to the mean vector $\mathbf{f}$ (Poisson distributed flows), Branston and Van Zuylen (1982), Maher (1983), but other experimental values can obviously be used.

By substituting in (3) the vector $\hat{\mathbf{f}}$ instead of $\mathbf{f}$, we obtain:

$$\hat{\mathbf{f}} = \hat{\mathbf{A}}\mathbf{t} + \boldsymbol{\eta} \tag{4}$$

where $\boldsymbol{\eta}$ is a random vector with mean equal to the bias $\boldsymbol{\delta}$ and dispersion matrix $\mathbf{W}$. As a particular case the random part of $\boldsymbol{\eta}$ could be ignored and vector $\boldsymbol{\delta}$ substituted to $\boldsymbol{\eta}$ in expression (4).

The problem is to find an estimator of the "true" O–D matrix $\mathbf{t}$ combining, in the best way in some sense, the information contained in the direct or model estimates $\hat{\mathbf{t}}$, in the flow observed on the network links, expressed by (1) and (4), and in the respective dispersion matrices.

3. THE GENERALIZED LEAST SQUARES ESTIMATOR FOR THE O–D MATRIX

The estimation problem defined in the previous section, in the more general case, can be seen as the research of an estimator for the parameters vector $\mathbf{t}$ of the linear model (1), subject to a system of stochastic equality constraints, system (4). Both stochastic systems have, in general, heteroscedastic and correlated residuals.

The two systems of equation given by (1) and (4) can be combined, thereby producing one system of linear stochastic equations that puts a vector of observations with dimensions $(n + m) \times 1$ in relation to the unknown parameters vector $\mathbf{t}$ and a random vector:

$$\begin{bmatrix} \hat{\mathbf{t}} \\ \hat{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n \\ \hat{\mathbf{A}} \end{bmatrix} \mathbf{t} + \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix}. \tag{5}$$

The mean of the random vector at the second member is :

$$\mathbf{E}(\boldsymbol{\epsilon}, \boldsymbol{\eta})' = (\boldsymbol{\mu}, \boldsymbol{\delta})'. \tag{6}$$

To express the dispersion matrix of the above random vector it is necessary to make some hypotheses about the assignment model adopted. Under the hypothesis of proportional assignment model, made in the previous section, it can be held that the components of the vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ are independent and therefore their dispersion matrix $\mathbf{B}$ is block diagonal:

$$\mathbf{B} = \begin{bmatrix} \mathbf{V} & \mathbf{O} \\ \mathbf{O} & \mathbf{W} \end{bmatrix}. \tag{7}$$

A possible extension to the equilibrium assignment case will be made at the end of the next section.

The generalized least squares estimator, or Aitken estimator, $\mathbf{t}^x$ of the vector of parameters $\mathbf{t}$ can be applied to the linear statistical model (5) with dispersion matrix of the random residuals given by (7).

The generalized least squares estimator, under the hypothesis that the random vector $(\boldsymbol{\epsilon}, \boldsymbol{\eta})'$ has zero mean, is the best linear unbiased estimator (BLUE) of the O–D matrix $\mathbf{t}$, i.e. the estimator of minimum variance in the class of all unbiased estimators linear in the vectors $\hat{\mathbf{t}}$ and $\hat{\mathbf{f}}$, Kendall and Stuart (1969), Judge *et al.* (1980). No distributional assumption for the random vector $(\boldsymbol{\epsilon}, \boldsymbol{\eta})$ is needed in the development of the following results. If, however, the random vector can be considered distributed according to a multivariate normal, as is the case when the O–D cells and flow values are big enough, the Aitken estimator coincides with the maximum likelihood estimator and therefore $\hat{\mathbf{t}}$ is the minimum variance estimator among all unbiased ones.

The Aitken estimator $\mathbf{t}^x$ is the solution of the quadratic programming problem:

$$\min_{\mathbf{t}} \begin{bmatrix} \hat{\mathbf{t}}-\mathbf{t} \\ \hat{\mathbf{f}}-\hat{\mathbf{A}}\mathbf{t} \end{bmatrix}' \begin{bmatrix} \mathbf{V}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}}-\mathbf{t} \\ \hat{\mathbf{f}}-\hat{\mathbf{A}}\mathbf{t} \end{bmatrix}. \tag{8}$$

From an intuitive point of view, the estimator $\mathbf{t}^x$ of the O–D matrix is the one of minimum weighted distance from the starting estimate $\hat{\mathbf{t}}$ that, once assigned to the network, gives rise to values of the flows of minimum weighted distance from the observed flows $\hat{\mathbf{f}}$. The weights are such as to make the distance of a component of $\mathbf{t}^x$ or of $\hat{\mathbf{A}}\mathbf{t}^x$ from the analogous component of $\hat{\mathbf{t}}$, or of $\hat{\mathbf{f}}$, the "shorter" the more dispersed are these component estimates so that the more likely it is that the observed value is far from the true one.

It is known that the true O–D matrix has non-negative cell values so that the objective function (8) should be minimized subject to non-negativity constraints, thus improving the statistical properties of $\mathbf{t}^x$. In the first stage of the analysis, however, the inequality constraints will be considered inactive so as to give the estimator $\mathbf{t}^x$ with its mean and dispersion matrix explicitly in terms of vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{f}}$ and of their dispersion matrices. The effects of the inequality constraints will be qualitatively evaluated later on.

In this case the Aitken estimator $\mathbf{t}^x$, because of the positive definiteness of the quadratic form in (8), can be obtained by equating to zero the first partial derivatives of (8) with respect to $\mathbf{t}$. The result is:

$$\mathbf{t}^x = (\mathbf{V}^{-1} + \hat{\mathbf{A}}'\mathbf{W}^{-1}\hat{\mathbf{A}})^{-1}\,(\mathbf{V}^{-1}\hat{\mathbf{t}} + \hat{\mathbf{A}}'\mathbf{W}^{-1}\hat{\mathbf{f}}). \tag{9}$$

It can be shown that the mean of $\mathbf{t}^x$ is:

$$\mathbf{E}(\mathbf{t}^x) = \mathbf{t} + (\mathbf{V}^{-1} + \hat{\mathbf{A}}'\mathbf{W}^{-1}\hat{\mathbf{A}})^{-1}\,(\mathbf{V}^{-1}\boldsymbol{\mu} + \hat{\mathbf{A}}'\mathbf{W}^{-1}\boldsymbol{\delta}) \tag{10}$$

and its dispersion matrix is:

$$\mathbf{D}(\mathbf{t}^x) = (\mathbf{V}^{-1} + \hat{\mathbf{A}}'\mathbf{W}^{-1}\hat{\mathbf{A}})^{-1}. \tag{11}$$

A different estimator $\mathbf{t}_1{}^x$ can be obtained by considering the observed flows and the true ones and therefore the constraint system deterministic. This is the case when the dispersion of the measured flows is supposed to be very small with respect to that of the O–D matrix starting estimate. The problem now is reduced to that of the estimation of the parameters vector $\mathbf{t}$ of the linear model (1) with heteroschedastic and correlated residuals, subjected to the system of linear constraints obtained by (3) crossing out the vector $\boldsymbol{\delta}$. The constrained least squares estimator for this problem is the solution of the quadratic programme:

$$\min_{\mathbf{t}} (\hat{\mathbf{t}}-\mathbf{t})'\mathbf{V}^{-1}(\hat{\mathbf{t}}-\mathbf{t}) \tag{12}$$

$$\hat{\mathbf{A}}\mathbf{t} = \hat{\mathbf{f}}$$

$$\mathbf{t} \geq \mathbf{0}.$$

Under the further assumption of inactivity of the inequality constraints, the solution to problem (12) can be obtained with the classic Lagrange multiplier technique:

$$\mathbf{t}_1{}^x = \hat{\mathbf{t}} + \mathbf{V}^{-1}\hat{\mathbf{A}}'(\hat{\mathbf{A}}\mathbf{V}^{-1}\hat{\mathbf{A}}')^{-1}\,(\mathbf{f} - \hat{\mathbf{A}}\hat{\mathbf{t}}). \tag{13}$$

The mean and dispersion matrix of $\mathbf{t}_1{}^x$, under the hypothesis of no random elements in f, are given by:

$$\mathbf{E}(\mathbf{t}_1{}^x) = \mathbf{t} + \boldsymbol{\mu} + \mathbf{V}\hat{\mathbf{A}}'(\hat{\mathbf{A}}\mathbf{V}^{-1}\hat{\mathbf{A}}')^{-1}(\boldsymbol{\delta} - \hat{\mathbf{A}}\boldsymbol{\mu}) \tag{14}$$

$$\mathbf{D}(\mathbf{t}_1{}^x) = \mathbf{V} - \mathbf{V}\hat{\mathbf{A}}'(\hat{\mathbf{A}}\mathbf{V}\hat{\mathbf{A}}')^{-1}\hat{\mathbf{A}}\mathbf{V}. \tag{15}$$

The estimator $\mathbf{t}_1{}^x$ can be compared with the maximum entropy estimator, Van Zuylen and Willumsen (1980), as both are minimum distance estimators, weighted according to the inverse of the dispersion matrix and "entropy" distance from a vector $\hat{\mathbf{t}}$ subject to the same linear constraints.

It must be said that both the estimators demand that the constraint vector **f** is consistent and therefore, as the flows measured usually are not consistent, a preventive "cleaning" is necessary. After this operation, however, nothing can guarantee that there is a feasible solution to problem (12) as the true vector **t** is not a solution of the constraint system if the vector $\boldsymbol{\delta}$ in (3) is not null.

In the same framework an equilibrium assignment model can also be used. The objective function (12) should be minimized instead of the other distance measures considered in the literature of O–D matrix estimation from counts under equilibrium assignment Nguyen (1982), Gur *et al.* (1980). In this case, however, it is very difficult, if at all possible, to express the statistical characteristics of the obtained estimates.

At this point something should be said about the relationship existing between the Aitken estimator presented in this paper and the Bayesian estimator proposed by Maher (1983).

Although there are strong formal similarities between them and their dispersion matrices, the two estimators are substantially different. First of all, the underlying theories, i.e. frequential and subjective interpretation of probability, are very different and so are the obtained results' interpretations. From a practical point of view, the dispersion matrix **V** in the Aitken estimator case is a starting O–D matrix estimator dispersion matrix, while in the Bayes estimator synthetizes the prior beliefs about the prior or "target" O–D matrix.

Furthermore, the Maher's Bayes estimator is valid under multinormal distribution assumption both for the prior O–D matrix distribution and for the counts' distribution while no distributional assumptions are required for the Aitken estimator. Finally, the results obtained in the two frameworks rapidly diverge when other assumptions are made, such as different prior or observation distributions and presence of inequality constraints.

4. SOME STATISTICAL PROPERTIES OF THE TWO ESTIMATORS

From the statistical point of view, it can be shown by using (11) and (15) that the difference between the dispersion matrix **V** of the starting estimate and the dispersion matrices of the two Aitken estimators (9) and (13), results in a positive definite matrix.

This means that the variance of every component of $\mathbf{t}^x$ and $\mathbf{t}_1{}^x$, is lower than that of the corresponding component of the starting estimator. In general, it is not possible to make a synthetic evaluation of this variance reduction. A rough idea can be obtained, however, by considering the deterministically constrained estimator $\mathbf{t}_1{}^x$ with homoschedastic and independent residuals, i.e. $\mathbf{V} = \sigma^2\mathbf{I}_n$. In this case, by (16), the following result is obtained:

$$\begin{aligned} \text{trace}\,[\mathbf{D}(t_1{}^x)] &= n\sigma^2 - \sigma^2\,\text{trace}\,[\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}] = \\ &= \sigma^2\,(n - m). \end{aligned} \tag{16}$$

This means that the percentual reduction of the sum of variances of $\mathbf{t}_1{}^x$ with respect to that of the starting estimator $\hat{\mathbf{t}}$ is proportional to the ratio between the number of constraint links m and the number n of O–D pairs to be estimated. In more general cases, however, the variance reduction will depend also on which links the flows were measured on, i.e. matrix $\hat{\mathbf{A}}$, and this can be used as a criterium for the constraint links selection.

On the other hand, it can be seen by (10) and (14) that both the stochastically and deterministically constrained estimators are biased if the direct or model estimator $\hat{\mathbf{t}}$ is biased and/or the assignment model is misspecified.

When this is the case, in spite of the above-mentioned variance reduction, the Aitken estimators are not "better" than the starting one in general. Their statistical "performances" must be compared with those of the starting estimator in order to evaluate if and by how much, the use of the information on the O–D matrix contained in the observed flows and in the assignment model improves the trip table estimation.

The criteria commonly used in statistics to compare estimators are those of mean square error (MSE or risk) and generalized mean square error (GMSE), both deriving from squared error loss functions in a decision theory framework, Judge *et al.* (1980).

The results of the preceding section giving biases and dispersion matrices of the two generalized least squares estimators allow the use of risk and GMSE criteria.

The comparison between the Aitken estimators and the direct or model ones must be carried out case by case; from a general point of view it can be said that when the starting estimator is unbiased and the assignment model is correctly specified the Aitken estimators are superior as results from the previous discussion about their variance reduction.

Furthermore, the Aitken estimators are superior to the starting ones under the GMSE criterium even in the most unfavourable case, i.e. unbiased director model estimator and misspecified assignment model, if the assignment distorsion $\boldsymbol{\delta}$ respects the inequality:

$$\boldsymbol{\delta}'(\hat{\mathbf{A}}\mathbf{V}\hat{\mathbf{A}}' - \mathbf{W})^{-1}\boldsymbol{\delta} \leq 1$$

in the stochastically constrained case, and:

$$\boldsymbol{\delta}'(\hat{\mathbf{A}}\mathbf{V}\hat{\mathbf{A}})^{-1}\boldsymbol{\delta} \leq 1$$

in the deterministically constrained case, Judge and Bock (1978).

Up to now the inequality constraints were considered to be inactive; if the non-negativity constraints in programmes (8) and (12) are active, the previous analysis is no longer strictly valid. Even if the general conclusions remain as such, it is not easy, or even possible, to express the results for such a complex case. From a general point of view, however, it can be shown that, by adding the inequality constraints if the direction of inequality is correct, as is certainly the case, both the risk and the GMSE of the two Aitken estimators are not worse than those previously obtained. The positive effect obtained decreases the further the true vector $\mathbf{t}$ is from the null vector. In some cases, it is also possible to substitute stronger inequality constraints to those of non-negativity, thereby further improving the characteristics of the Aitken estimator. If, for example, the journeys between the various zone pairs observed during the sample survey are ordered in a vector $\mathbf{c}$, the non-negativity constraints can be substituted by the system of inequality constraints: $\mathbf{t} \geq \mathbf{c}$. In order to obtain approximate expressions of the bias and of the dispersion matrix, the results of the deterministically constrained estimator with the equality constraints corresponding to the active inequality constraints can be applied. This is the same as supposing that the active constraints would result as such whatever vector $\hat{\mathbf{t}}$ resulted from the sampling experiment that determines it.

5. A QUANTITATIVE EVALUATION OF THE ESTIMATOR'S PERFORMANCE ON A SMALL ARTIFICIAL NETWORK

In this paper, up to now, it has been assumed that the dispersion matrices $\mathbf{V}$ and $\mathbf{W}$ of the O–D estimator $\mathbf{t}$ and of the observed flows vector $\hat{\mathbf{f}}$ were known exactly.

Obviously in practice this is not the case: only estimates of these matrices are available. In all the expressions contained in the previous sections, therefore, the estimated matrices $\hat{\mathbf{V}}$ and $\hat{\mathbf{W}}$ should be substituted to the true ones $\mathbf{V}$ and $\mathbf{W}$ and the resulting estimators are in fact "estimated generalized least squares" or EGLS estimators.

For EGLS estimators it is not possible to establish exact results regarding risk and GMSE and therefore the comparison of their performances for finite samples. The results of Monte Carlo simulations, carried out by many researchers, indicate that EGLS estimators are generally superior to the least squares ones considering homoschedastics and independent residual when this is not the case. To verify these results and to obtain a magnitude order of the variance reduction of the Aitken estimator proposed a small simulation exercise was carried out.

The network of Fig. 1 was considered with five centroids into which trips can end and start; the number N_i of potential travellers is reported in each centroid; the "true" O–D matrix is also shown in the same figure.

The results of 100 O–D surveys were simulated a half with a sampling rate of 5% and the other half with a 10% sampling rate. It was assumed that in each centroid "i" a number n_i of travellers were independently extracted (as in the case of a plate number survey or a nominal home survey) so that:

$$n_i = \alpha \cdot N_i.$$

For each sampled traveller starting his trip in zone "i" the destination "j" was simulated as the result of a multinomial experiment with probability π_{ij} given by:

$$\pi_{ij} = \frac{t_{ij}}{N_i}$$

where t_{ij} is the true number of travellers going from zone "i" to zone "j".

The O–D matrix starting estimates $\hat{\mathbf{t}}$ were obtained by using the direct estimator:

$$\hat{t}_{ij} = n_{ij} \cdot \frac{1}{\alpha}$$

where n_{ij} is the number of trips between zones "i" and "j" resulting from the sampling experiment and $1/\alpha$ the grossing up factor.

It is well known that in this case the sampling variances and covariances of the obtained estimates are, Cochran (1963):

$$\text{var}\,(\hat{t}_{ij}) = \frac{N_i^2}{n_i}\,\pi_{ij}(1 - \pi_{ij}) \tag{17}$$

O \ D	A	B	C	D	E
A	–	100	300	400	100
B	50	–	125	225	50
C	300	100	–	700	700
D	500	250	500	–	1000
E	300	225	225	600	–

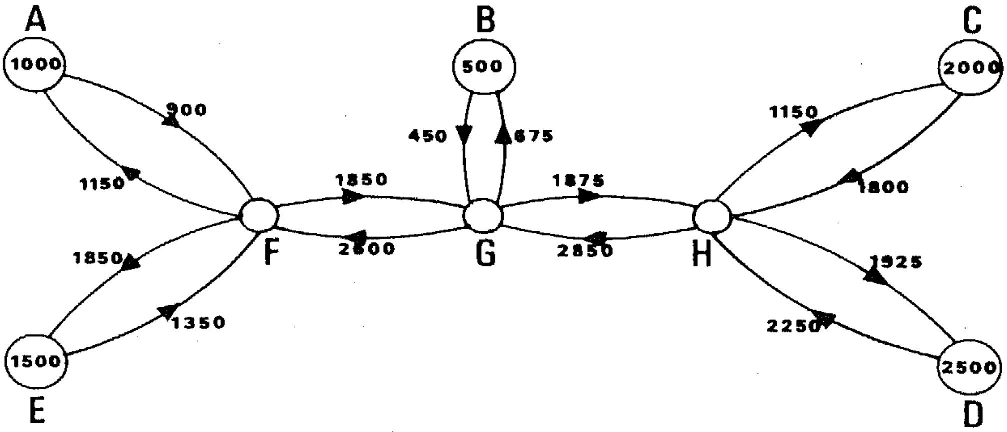

Fig. 1. Test network with O/D matrix and flows.

and

$$\text{Cov}(\hat{t}_{ij}, \hat{t}_{ik}) = -\frac{N_i^2}{n_i}\pi_{ij}\pi_{ik} \tag{18}$$

$$\text{Cov}(\hat{t}_{ij}, \hat{t}_{lm}) = 0. \tag{19}$$

In the above discussion the hypothesis was implicitly made that, because of the very small sampling fraction used, the finite population effects were negligible.

For each simulated survey many GLS estimators deterministically constrained to the flow observed on four links, i.e. links F–G, G–F, G–M and H–G, and on all independent links were calculated by (13).

In particular, with the purpose of quantifying the effects of increasingly less exact expressions for the starting estimator dispersion matrix, for each simulation experiment the following O–D estimates were carried out:

(1) estimation of the vector $\mathbf{t}_1^x$ using the true dispersion matrix $\mathbf{V}$ as given by (17)–(19);

(2) estimation of the vector $\mathbf{t}_2^x$ using an estimated dispersion matrix $\hat{\mathbf{V}}$ obtained by (17)–(19) substituting the estimated probabilities p_{ij} to the true ones π_{ij};

(3) estimation of the vector $\mathbf{t}_3^x$ using a dispersion matrix given by the main diagonal of the true dispersion matrix $\mathbf{V}$;

(4) estimation of the vector $\mathbf{t}_4^x$ using a dispersion matrix given by the main diagonal of the estimated dispersion matrix $\hat{\mathbf{V}}$;

(5) estimation of the vector $\mathbf{t}_5^x$ using a diagonal dispersion matrix proportional to the direct estimate $\hat{\mathbf{t}}$;

(6) estimation of the vector $\mathbf{t}_6^x$ using an identity dispersion matrix;

(7) estimation of the vector $\mathbf{t}_7^x$ of maximum entropy.

Because the starting estimator $\hat{\mathbf{t}}$ is unbiased and the network structure is such that no alternative routes between every O–D pair exist ($\boldsymbol{\mu} = \boldsymbol{\delta} = \mathbf{0}$) all the GLS estimators $\mathbf{t}_1^x - \mathbf{t}_6^x$ are unbiased, as it results from (14). To make unbiased also the maximum entropy estimator, in estimating $\mathbf{t}_7^x$ the further constraint of respecting the total number of trips was added.

The mean square errors (MSE) were then calculated† for all the estimators, separately for the 5 and 10% sampling fractions. In Table 1 the MSE values, for the 10% sampling fraction are reported with the value relative to the direct estimator $\hat{\mathbf{t}}$ assumed as a comparison basis.

Table 1. Comparison of different constrained estimators mean square errors

ESTIMATOR	MSE CONSTRAINED ON 4 LINKS	MSE CONSTRAINED ON ALL LINKS
$\hat{t}$	100	100
t_1^x	82	52
t_2^x	83	53
t_3^x	88	53
t_4^x	88	53
t_5^x	92	62
t_6^x	93	62
t_7^x	112	68

†In this case the MSE was given by the sum of the sampling variances relative to each component of the various estimators.

The results relative to the 5% sampling fraction are not shown because they mainly coincide with those reported in Table 1 when expressed as a percentage of the direct estimators MSE. This means that the variance reduction in absolute terms will be the bigger, the lower the sampling fraction is.

Although the reported results are very partial, on the basis of Table 1 the following considerations can be made:

(a) the obtained estimates get worse the further the dispersion matrix used is from the true one as is to be expected;

(b) the variance reduction obtained by using estimated dispersion matrices is very close to that obtained by using the true dispersion matrices;

(c) the variance reduction obtained with the estimator $\mathbf{t}_1^x$ is approximately equal to the ratio between the number of constraint links and O–D pairs as predicted in the previous section by (16);

(d) the maximum entropy estimator's MSE is bigger than that of GLS estimators, even when very heavy approximations for the dispersion matrix are used;

(e) the approximation of diagonal dispersion matrix can be an acceptable one, especially with many constraint links.

CONCLUSIONS

In this paper a generalized least squares estimator of the O–D matrix was proposed combining trip table's direct or model estimators with traffic counts via an assignment model. Two cases were considered: a more general one, in which the estimator was stochastically constrained to the observed flows considered to be random variables because of measurement errors and temporal fluctuations, and another in which the estimator was deterministically constrained to the observed flows. Analytical expressions were given for both estimators in the case of inactive inequality constraints. Under the same hypothesis their means and dispersion matrices were obtained in terms of the possible bias in the starting O–D estimates and of the assignment model misspecification.

The variance of each O–D cell obtained with the GLS estimators was proved to be lower than that obtained with direct or model estimators.

It was noted, however, that the GLS estimators can be biased when the starting estimators are so and/or the assignment model is misspecified. For these reasons the GLS estimators' statistical performances must be compared with those of classical O–D matrix estimators under criteria of risk and generalized mean square error to see if, and by how much, the information about the O–D matrix contained in the observed flows and in the assignment model improves the estimation.

Risk and GMSE criteria can be applied using the bias and dispersion matrix expressions reported in the paper.

From a general point of view, it can be said that the Aitken estimators are superior under both criteria when they are unbiased. Limits for the assignment model specification error were also set up in the most unfavourable case of unbiased starting estimator. When these limits are respected Aitken estimators are superior under the more restrictive GMSE criterium.

Finally a small simulation exercise was carried out to get a rough idea about the effects of substituting more or less approximate dispersion matrices to the true ones in the previous results and to compare the performances of the proposed estimator with those of the maximum entropy one.

It resulted that the substitution of an estimated dispersion matrix in place of the true one produced only a slight worsening of the estimator's characteristics and that all the Aitken estimators considered had a mean square error lower that the maximum entropy estimator.

REFERENCES

Bell M. G. H. A. (1983) The estimation of an origin–destination matrix from traffic counts. *Transpn Sci.* 10, 198–217.

Cascetta E. (1983) Alcune considerazioni sui metodi di stima della matrice Origine–Destinazione. Transportation Institute, University of Naples, Internal Report.

Cochran W. G. (1963) *Sampling Techniques*, 2nd Edn. Wiley, New York.

Dial R. B. (1971) A probabilistic multipath traffic assignment model which obviates path enumeration. *Transpn Res.* **5**, 83–111.

Florian M. and Nguyen S. (1976) An application and validation of equilibrium trip assignment methods. *Transpn Sci.* **10**, 374–389.

Gur J., Hutsebout O., Kurth D., Clark M. and Castilla J. (1980) Estimation of an Origin–Destination trip table based on observed link volumes and turning movements, Volume 1. *Technical Report, FHWA, Report No. RD*-801035. U.S. Dept. of Transportation Washington D.C..

Judge G. G. and Boch M. E. (1978) *The Statistical Implications of Pre-Test and Stein-Rule Estimators in Econometrics*. North Holland, Amsterdam.

Judge G. G., Griffiths W. E., Hill R. C. and Lee T. C. (1980) *The Theory and Practice of Econometrics*. Wiley, New York.

Kendall M. and Stuart A. (1969) *The Advanced Theory of Statistics* Vol. 1, 3rd Edn. Griffin, London.

Kendall M. and Stuart A. (1973) *The Advanced Theory of Statistics* Vol. 2, 3rd Edn. Griffin, London.

Maher M. J. (1983) Inferences on trip matrices from observations on link volumes: a Bayesian statistical approach, *Transpn Res.* **17B**.

Nguyen S. (1978) A unified approach to equilibrium methods of traffic assignment. In *Traffic Equilibrium Methods*. Edited by M. Florian. Vol. 118, pp. 148–182, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin.

Nguyen S. (1982) Estimating origin–destination matrices from traffic flows. 1*st Course in Transportation Planning Model*, ICTS, Amalfi.

Van Vliet D. and Willumsen L. G. (1981) Validation of the ME2 model for estimating trip matrices from traffic counts, *Int. Symp. on Transportation and Traffic Theory*, University of Toronto, Toronto, Canada.

Van Zuylen J. H. and Willumsen L. G. (1980) The most likely trip matrix estimated from traffic counts. *Transpn Res.* **14B**, 281–293.

Van Zuylen J. H. and Branston D. M. (1982) Consistent link flow estimation from counts. *Transpn Res.* **16B**, 473–476.

Willumsen L. G. (1978) Estimation of an O–D matrix from traffic counts: a review. Institute for Transport Studies, Leeds University.

Yates F. (1981) *Sampling Methods for Censurer and Surveys* 4th Edn. Griffin, London.